# Hilbert Space

A **Hilbert space** is a fundamental concept in functional analysis and quantum mechanics, named after the German mathematician David Hilbert. It is a type of abstract vector space that extends the idea of Euclidean space (used in geometry) to infinite-dimensional spaces.

## Pre-requisite Theoretical Concepts

Let us brush up the following concepts first before we fully dive into the understanding of Hilbert Space.

### Vector Space

**(Definition)** A set is called an vector space, if it is equipped with an addition and a scalar multiplication such that the following properties are satisfied for all and .

1. and
2. There is an element such that
3. There is an element called such that
4. and

There are many different vector spaces that will pop up in data science:

1. Most important: , the that consists of tuples of real numbers. The space of complex vectors is less prominent but may occur as well.
2. As important: *Spaces of functions!* The set of all functions from some set into the real numbers form a vector space as well. Such a space may be used to model a set of decision functions that we want to build to predict an output for given data . If we want to predict more than just one number for a data point we consider the space of functions , and those functions form a vector space as well.

### Inner Product

**(Definition)** Let be a real vector space. An inner product on is a map with the following properties:

1. For all and , we have linearity in the first argument, i.e.,
2. For all , we have symmetry i.e.,
3. For all , we have positive definiteness i.e.,

**(Remark)**

1. Properties (i) and (ii) together imply that is also linear in the second argument. Hence, an inner product is a so-called bilinear map.
2. For vector spaces over , we require ***sesquilinearity***, i.e.,

and

for and .

**(Example)**

1. On , there is the standard inner product, also called dot product . The inner product can also be written using matrix-vector multiplication as .
2. On the space of functions , we can define the so-called -inner product
3. In the spirit of the first point, we can define for any matrix a bilinear map

This map is symmetric when is so and is an inner product when is positive definite, i.e., when for .

**It is of fundamental importance that *inner products induce norms*, i.e., for any inner product we can define**

**(Theorem) Cauchy – Schwartz Inequality for Inner Product Spaces**

Let be a vector space with inner product . Then, for all , we have

Proof.

Without loss of generality we may assume, . Then, for , we have:

For any , the expression represents the inner product of a vector with itself, which is always non-negative.

Now, since , the discriminant of this quadratic must be less than or equal to zero (for it to not have any real roots or to have a double root). Therefore,

This gives us the required expression:

### Metric

**(Definition)** Let be a set. A metric on is a map

with the following properties:

1. For all , it holds that [*symmetry*]
2. It holds that , exactly if [*positive definite*]
3. For all , it holds that all [*triangle inequality*]

Similar to the case of norm, we can define convergence of sequences in metric spaces. We say with respect to the metric if .

**(Example)**

Let be a unit circle in , i.e., the set .

Since is a subset of the vector space , we could measure the distance of by which would be the distance we measure with a ruler. Alternatively, we could also measure the distance by the angle between .

More generally, the angel between two vectors in is given by

And this angle does indeed define a metric on the unit sphere .

**(Remarks)**

**Similar to how inner product induces norms, it is also true that norms induce metrics**: if the set is also a vector space and is a norm on this space, then:

is in fact a metric.

**Metrics are often used when the underlying space is not a vector space.**

Now, we have:

which means, the metric is the most general notation. Note that a metric need not be induced by a norm. Consider, e.g., the discrete metric:

Clearly there are , for which with , i.e., the homogeneity property fails.

### Metric Space

A metric space is a set equipped with a metric. More formally, a metric space is a pair , where:

* is a set (the elements of which can be points, vectors, or other objects)
* is a metric on , i.e., a function that satisfies the above properties of a metric

Thus, a metric space is the complete structure that includes both the set and the distance function .

**(Example)**

Consider the set and the standard absolute value metric .

* The metric here is , which tells us the distance between any two real numbers and .
* The metric space is the pair , where is the set of real numbers, and is the metric.

**(Remark)**

Let be a set equipped with a metric .

1. For and we define

and call the *open ball* of radius around or the -neighborhood of .

1. A set is called *open*, if for all there exists an (which is allowed to depend on ) such that
2. A set is called *closed*, if its complement set is open.

**(Remark)**

Let be a metric space.

1. If some given subset is open or not depends on the set which contains .
2. There are sets which are neither open, nor closed – think of the half-open interval as subset of . On the other hand, there are sets which are both open and closed, called “*clopen*”. For instance, both the set of the original metric space and the empty set are both open and closed.

**(Remark)**

In a metric space , a set is closed if and only if the limit of every convergent sequence with , one has . Put differently, convergent sequences in cannot leave the set.

**(Example)**

We consider the real line with the metric .

* The intervals are open. (Given some , can you find an such that ?)
* The intervals are closed. (Which follows from the fact that if and , then ).
* The intervals and are neither closed, nor open, while and the empty set are both open and closed.

## Foundation of Hilbert Space

## Key Features of Hilbert Space

The key features of a Hilbert space are:

1. **Vector space**: It is a vector space, meaning it has elements (vectors) that can be added together and scaled by numbers (scalars), following certain rules.
2. **Inner product**: Each Hilbert space has an inner product, which is a function that takes two vectors and returns a scalar (usually a real or complex number). This inner product provides a notion of angle and length, allowing concepts like orthogonality and distance between vectors.
3. **Completeness**: Hilbert spaces are complete with respect to the norm induced by the inner product. This means that every Cauchy sequence (a sequence where the vectors get arbitrarily close to each other) in the Hilbert space has a limit that also lies within the space.
4. **Infinite dimensions**: While Hilbert spaces can be finite-dimensional (like regular Euclidean space), they are most often used in contexts where the space is infinite-dimensional, such as in quantum mechanics or the theory of partial differential equations.

## Applications of Hilbert Space

* **Quantum mechanics**: In quantum theory, the state of a physical system is represented as a vector in a Hilbert space. Observables (like position, momentum, and energy) are represented by operators acting on this space.
* **Fourier analysis**: Hilbert spaces are used to represent functions as sums of basic waveforms, which is a key idea in Fourier series and transforms.
* **Signal processing**: Functions or signals can be represented in Hilbert spaces, where operations like filtering and signal reconstruction are defined in terms of the inner product.